

SPACE-TIME CODE DESIGN FOR THREE-TRANSMIT-ANTENNA SYSTEMS

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ABSTRACT

Fully-diverse constellations, i.e., a set of unitary matrices whose pairwise differences are nonsingular, are useful in multi-antenna communications especially in multi-antenna differential modulation, since they have good pairwise error properties. Recently, group theoretic ideals, especially *fixed-point-free (fpf)* groups, have been used to design fully-diverse constellations of unitary matrices. Here we give a systematic method to design space-time codes which are appropriate for three-transmit-antenna differential modulation. The structure of the code is motivated by the Lie group $SU(3)$. The code has a fast decoding algorithm using sphere decode. The diversity product of the code can be easily calculated and simulated performance show that the code is better than the group-based codes [1] especially at high rates and is as good as the elaborately-designed non-group code [1].

1. INTRODUCTION

It is well known in theory that multiple antennas can greatly increase the data rate and the reliability of a wireless communication link in a fading environment. In practice, however, one needs to devise effective space-time transmission schemes. This is particularly challenging when the propagation environment is unknown to the sender and the receiver, which is often the case for mobile applications when the channel changes rapidly.

A differential transmission scheme called *differential unitary space-time modulation* was proposed in [2], which is well-tailored for unknown continuously varying Rayleigh flat-fading channels. The signals transmitted are unitary matrices. In this scheme the probability of error of mistaking one signal S_i for another $S_{i'}$, at high SNR, is proved to be inversely proportional to $|\det(S_i - S_{i'})|$. Therefore the quality of the code is measured by its *diversity product*

$$\xi_C = \frac{1}{2} \min_{S_i \neq S_{i'} \in C} |\det(S_i - S_{i'})|^{\frac{1}{M}} \quad (1)$$

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where M is the number of transmit antennas and C is the set of all possible signals. The design problem is thus the following: "Given the number of transmitter antennas, M , and the transmission rate, R , find a set C of $L = 2^{MR}$ $M \times M$ unitary matrices, such that the minimum of the absolute value of the determinant of their pairwise differences is as large as possible."

The space-time code designs for three-transmit-antenna system are rare. Till now, some group-based codes and non-group codes are proposed in [1]. The group-based codes, mainly the G_{mr} codes and diagonal codes, are rare and do not have good performances for high rates. The design of the non-group codes are very difficult and the decoding of both codes needs exhaustive search. In this paper, we proposed design methods for three-transmit-antenna systems. The codes are motivated by the Lie group $SU(3)$. The reasons of analyzing the Lie group $SU(3)$ are as follows.

The design problem, as just stated, appears to be intractable since first the signal set and the cost function are non-convex and second, the size of the problem can be huge, especially at high data rates. Therefore, in [1, 3], it was proposed to enforce a group structure on the constellation. This has the advantages of simplifying the diversity product and easy decoding [1, 3]. In [1], all finite fully-diverse constellations that form a group are classified. And also, in [3], it is proved that the only fpf infinite Lie groups are $U(1)$, the group of unit-modulus scalars, and $SU(2)$, the group of unit-determinant 2×2 unitary matrices. However, no good constellations are obtained for very high rates from the finite fpf groups, and constellations based on $U(1)$ and $SU(2)$ are constrained to one and two-transmit-antenna systems. As mentioned in [4], to get high rate constellations which work for systems with more than 2 transmit antennas, we relax the fpf condition by considering Lie groups of rank 2. (The rank of a Lie group equals the maximum number of commuting basis elements of its Lie algebra and it can be shown that fpf groups have rank 1.) There are three of them: the Lie group of unit-determinant 3×3 unitary matrices $SU(3)$, the Lie group of 4×4 unitary, symplectic matrices $Sp(2)$, and one exceptional Lie group G_2 . Constellations based on $Sp(2)$, which can be regarded as an extension of the Alamouti's

scheme [5], are designed in [4] and simulation results show that they have good performance. In this work, we analyze $SU(3)$, which gives us 3×3 constellations.

The code we propose in this paper has a simple formula from which the diversity product of it can be calculated in a fast way. Necessary conditions for the full-diversity of the code are also proved. Our conjecture is that they are also sufficient conditions. Simulation results show that the codes have better performances than the group-based codes [1] especially at high rates and are as good as the elaborately-designed non-group codes [1]. Another exceptional feature of the code is that it has a fast decoding algorithm based on complex sphere decoding [6].

1.1. Differential Unitary Space-time Modulation

Consider a wireless communication system with M transmit antennas and N receive antennas. The channel is used in blocks of M transmissions (for more on this model, see [7, 8]), the system equations of block τ can be written as:

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + V_\tau$$

Here, S denotes the $M \times M$ transmitted signal with s_{tm} the signal sent by the m th transmit antenna at time t . H is the $M \times N$ complex-valued propagation matrix, which is unknown to both the transmitter and the receiver, and h_{mn} is the propagation coefficient between the m th transmit antenna and the n th receive antenna and has an iid $\mathcal{CN}(0, 1)$ distribution. V is the $M \times N$ noise matrix with v_{tn} , the noise at the n th receive antenna at time t , iid $\mathcal{CN}(0, 1)$ distribution. X is the $M \times N$ received signal matrix. The transmitted power constraint is $\sum_{m=1}^M \mathbb{E} |s_{tm}|^2 = 1$, $t = 1, \dots, M$ so ρ represents the expected SNR at each receive antenna.

In differential modulation, the transmitted matrix S_τ at block τ equals to the product of the previously transmitted matrix and a unitary data matrix V_{z_τ} taken from our signal set \mathcal{C} . In other words, $S_\tau = V_{z_\tau} S_{\tau-1}$ where $S_0 = I_M$. The transmission rate is $R = \frac{1}{M} \log_2 L$, where L indicates the cardinality of our code. Further assume that the propagation environment keeps approximately constant for $2M$ consecutive channel uses, that is, $H_\tau \approx H_{\tau-1}$, we may get the fundamental differential receiver equations [9]

$$X_\tau = V_{z_\tau} X_{\tau-1} + W'_\tau \quad (2)$$

where $W'_\tau = W_\tau - V_{z_\tau} W_{\tau-1}$. We can see that the channel matrix H does not appear in (2). This implies that differential transmission permits decoding without knowing the channel information. The ML decoder of z_τ is given by

$$\hat{z}_\tau = \arg \max_{l=0, \dots, L-1} \|X_\tau - V_l X_{\tau-1}\| \quad (3)$$

It is shown in [2] that, at high SNR, the pairwise probability of error (of transmitting V_l and erroneously decoding $V_{l'}$) has an upper bound that is inversely proportional to the diversity product of the code.

2. PARAMETRIZATION OF $SU(3)$

Definition 1 [10] $SU(n)$ is the group of complex $n \times n$ matrices obeying $U^*U = UU^* = I_n$ and $\det U = 1$.

From the definition, $SU(n)$ is the group of complex $n \times n$ unitary matrices with determinant 1. It is also known that $SU(n)$ is a compact, simply-connected Lie group of dimension $n^2 - 1$ and rank $n - 1$. Since we are most interested in the case of rank 2, here we focus on $SU(3)$, which has dimension 8. The following theorem on the parametrization of $SU(3)$ is proved.

Theorem 1 (Parametrization of $SU(3)$) Any matrix U belongs to $SU(3)$ if and only if it can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \sqrt{1-|\alpha|^2} & 0 & \bar{\alpha} \\ 0 & 1 & 0 \\ -\alpha & 0 & \sqrt{1-|\alpha|^2} \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & 1 \end{bmatrix} \quad (4)$$

where $\Phi, \Psi \in SU(2)$. α is a complex scalar with $|\alpha| \leq 1$.

α actually equals the determinant of the sub-matrix of U by deleting the first row and the first column. This theorem indicates that any matrices in $SU(3)$ can be written as a product of three 3×3 unitary matrices which are basically $SU(2)$. (They are actually 3×3 unitary representations of $SU(2)$ by adding an identity block.) Now let's look at the number of degrees of freedom in U . Since $\Phi, \Psi \in SU(2)$, there are 6 degrees of freedom in them. Together with the 2 degrees of freedom in α , the dimension of U is 8, which is exactly the same as that of $SU(3)$. Based on (4), we can parameterize matrices in $SU(3)$ by entries of Φ, Ψ and α , that is, any matrix in $SU(3)$ can be identified with a 3-tuple (Φ, Ψ, α) . There is also an interesting symmetry in (4). The i th matrix has an identity block at the (i, i) entry.

3. $SU(3)$ CODE DESIGN

From (4) we can see that for any $U_1(\Phi_1, \Psi_1, \alpha_1), U_2(\Phi_2, \Psi_2, \alpha_2)$ in $SU(3)$, if any two elements of the 3-tuples are identical, the difference matrix is singular. This is because that each of the matrices in (4) has an identity block which results in that the difference of any two matrices of the kind has a zero block. That is, the identity entries spoil the full diversity of the sets. Therefore, we need to replace the identity entries. Note that the U in (4) can also be written as

$$\begin{bmatrix} e^{j\theta} & 0 & 0 \\ 0 & \phi_{11} & \phi_{12}e^{-j\theta} \\ 0 & -\bar{\phi}_{12} & \bar{\phi}_{11}e^{-j\theta} \end{bmatrix} \begin{bmatrix} \sqrt{1-|\alpha|^2}e^{-j(\theta+\xi)} & 0 & \bar{\alpha}e^{-j(\theta-\xi)} \\ 0 & 1 & 0 \\ -\alpha e^{j(\theta-\xi)} & 0 & \sqrt{1-|\alpha|^2}e^{j(\theta+\xi)} \end{bmatrix} \begin{bmatrix} \psi_{11}e^{j\xi} & \psi_{12}e^{j\xi} & 0 \\ -\bar{\psi}_{12} & \bar{\psi}_{11} & 0 \\ 0 & 0 & e^{-j\xi} \end{bmatrix}$$

Another thing is that since the Lie group is not pf, we cannot use all the 8 degrees of freedom to get a fully-diverse

code. Here, we want to sample the Lie group in an appropriate way to obtain fully-diverse subsets. Therefore, to simplify the structure, we set the middle matrix to be I_3 and discuss sets of matrices which are products of 2 $SU(2)$ matrix representations. Note also that the $e^{-j\theta}$ in the last column of the first matrix and $e^{j\xi}$ in the first row of the third matrix are used to make the matrices determinant 1. However, in differential unitary space-time code design, we only need the signal matrix to be unitary. Therefore, we can further simplify the structure by abandoning the restriction that each of the matrices are unit determinant. Define

$$A_{(p,q,\theta)} = \begin{bmatrix} e^{j\theta} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{2\pi j \frac{p}{P}} & \frac{1}{\sqrt{2}}e^{2\pi j \frac{q}{Q}} \\ 0 & -\frac{1}{\sqrt{2}}e^{-2\pi j \frac{q}{Q}} & \frac{1}{\sqrt{2}}e^{-2\pi j \frac{p}{P}} \end{bmatrix}$$

$$B_{(r,s,\xi)} = \begin{bmatrix} \frac{1}{\sqrt{2}}e^{2\pi j \frac{r}{R}} & \frac{1}{\sqrt{2}}e^{2\pi j \frac{s}{S}} & 0 \\ -\frac{1}{\sqrt{2}}e^{-2\pi j \frac{s}{S}} & \frac{1}{\sqrt{2}}e^{-2\pi j \frac{r}{R}} & 0 \\ 0 & 0 & e^{-j\xi} \end{bmatrix}.$$

The following codes are obtained.

$$\mathcal{C} = \left\{ A_{(p,q,\theta)}^{(2)} B_{(r,s,\xi)}^{(2)} \mid \begin{array}{l} p \in [0, P), q \in [0, Q), \\ r \in [0, R), s \in [0, S) \end{array} \right\} \quad (5)$$

Since the channel is used in blocks of 3 transmissions, the rate of the code is $R = \frac{1}{3} \log_2(PQRS)$.

The code in (5) is not a subset of the Lie group $SU(3)$ any more since the determinant of the matrices is now $e^{j(\theta-\xi)}$ which is not 1 in general. However, the matrices in the codes are still unitary matrices. Since any matrix in the code is a product of two unitary matrices, we call it the AB code. In the following section, we will see that this handy structure results in a fast decoding algorithm.

A necessary condition for full diversity is that both the sets $\{A_{(p,q,\theta)}\}$ and $\{B_{(r,s,\xi)}\}$ are fully-diverse. The θ and ξ need to be chosen carefully. By setting $\theta = 2\pi \left(\pm \frac{p}{P} \pm \frac{q}{Q} \right)$ and $\xi = 2\pi \left(\pm \frac{r}{R} \pm \frac{s}{S} \right)$, it is easy to see that when $\gcd(P, Q) = \gcd(R, S) = 1$, the two sets are fully-diverse.

Theorem 2 (Calculation of the diversity products) For $U_1(p_1, q_1, r_1, s_1, \theta_1, \xi_1), U_2(p_2, q_2, r_2, s_2, \theta_2, \xi_2) \in \mathcal{C}$,

$$|\det(U_1 - U_2)| = 2|\text{Im}[(\Theta_1 - \bar{\Theta}_1 w)(\bar{\Theta}_2 - \Theta_2 x)]|$$

where

$$x = e^{2\pi j \left(\frac{p_1 - p_2}{2P} - \frac{q_1 - q_2}{2Q} \right)} \cos 2\pi \left(\frac{p_1 - p_2}{2P} + \frac{q_1 - q_2}{2Q} \right)$$

$$w = e^{2\pi j \left(-\frac{r_1 - r_2}{2R} - \frac{s_1 - s_2}{2S} \right)} \cos 2\pi \left(\frac{r_1 - r_2}{2R} - \frac{s_1 - s_2}{2S} \right),$$

and $\Theta_1 = e^{j(\pm \frac{p_1 - p_2}{2P} \pm \frac{q_1 - q_2}{2Q})}$, $\Theta_2 = e^{j(\pm \frac{r_1 - r_2}{2R} \pm \frac{s_1 - s_2}{2S})}$. $\text{Im}[c]$ indicates the imaginary part of the complex scalar c .

We see that x, w, Θ_1, Θ_2 only depend on the differences $\delta_p = p_1 - p_2, \delta_q = q_1 - q_2, \delta_r = r_1 - r_2, \delta_s = s_1 - s_2$. That is, the determinant of any difference matrix can be written as a function of $\delta_p, \delta_q, \delta_r, \delta_s, \Delta(\delta_p, \delta_q, \delta_r, \delta_s)$. From (1), the diversity products of the codes equals

$$\min |2\text{Im}[(\Theta_1 - \bar{\Theta}_1 w)(\bar{\Theta}_2 - \Theta_2 x)]|^{1/3}/2.$$

The minimum is over $(\delta_p, \delta_q, \delta_r, \delta_s) \neq (0, 0, 0, 0)$. Since $\delta_p, \delta_q, \delta_r, \delta_s$ can take on $2P - 1, 2Q - 1, 2R - 1, 2S - 1$ possible values respectively, we only need to calculate the determinants of $(2P - 1)(2Q - 1)(2R - 1)(2S - 1) - 1 < 16PQRS = 16L$ difference matrices instead of $L(L - 1)/2$ ones in the general case, where L is the cardinality of the code. Actually, instead of $16L$, less than $8L$ calculations is enough since $|\Delta(\delta_p, \delta_q, \delta_r, \delta_s)| = |\Delta(-\delta_p, -\delta_q, -\delta_r, -\delta_s)|$. Therefore, the computational complexity is greatly reduced especially for codes of high rates.

Theorem 3 (Necessary conditions for fully-diverse) A necessary condition for the code to be fully-diverse is $\gcd(P, Q) = \gcd(R, S) = 1$ and at most one of P, Q, R, S is even.

Our conjecture is that the condition is also sufficient.

4. DECODING OF THE $SU(3)$ CODE

Matrices in the code is a product of two basically $U(2)$ matrices $A_{(p,q,\theta)}$ and $B_{(r,s,\xi)}$ and they have orthogonal design structure with PSK elements. This handy property can be used to get linear-algebraic decoding, which means that the receiver can be made to form a system of linear equations of the unknowns.

The ML decoder for differential USTM given in (3) is equivalent to

$$\arg \min_{p,q,r,s} \|X_\tau - A_{(p,q,\theta)} B_{(r,s,\xi)} X_{\tau-1}\|_F^2$$

$$= \arg \min_{p,q,r,s} \|A_{(p,q,\theta)}^* X_\tau - B_{(r,s,\xi)} X_{\tau-1}\|_F^2$$

Since A and B have orthogonal design structure, the decoder can be written equivalently as

$$\arg \min_{p,q,r,s} \|\mathcal{H}\mathcal{X}\|_F^2$$

where \mathcal{H} is a $3N \times 6$ complex matrix which only depends on

$$X_\tau \text{ and } X_{\tau-1} \text{ and } \mathcal{X} = \begin{bmatrix} e^{-j\theta}, e^{2\pi j \frac{p}{P}}, e^{-2\pi j \frac{q}{Q}}, e^{-j\xi}, e^{2\pi j \frac{r}{R}}, e^{2\pi j \frac{s}{S}} \end{bmatrix}^t$$

is the vector of unknowns. Since $\mathcal{H}\mathcal{X}$ is linear in the unknown vector \mathcal{X} , we can use sphere decoding for complex channels proposed in [6] with slight modification. The only difference here is that $e^{-j\theta}$ and $e^{-j\xi}$ are not independent unknown signals but are determined by p, q and r, s . Therefore, in the sphere decoding, instead of searching over the

intervals for $e^{-j\theta}$ and $e^{-j\xi}$, we calculate their values by the choices of p, q and r, s respectively based on the formulas $e^{2\pi j(\pm \frac{p}{P} \pm \frac{q}{Q})}$ and $e^{2\pi j(\pm \frac{r}{R} \pm \frac{s}{S})}$. Since sphere decoding has an average complexity that is cubic in the code rate and the dimension of the system and at the same time achieves the ML results, we have a fast ML decoding algorithm for the code.

5. SIMULATION RESULTS

In this section, the performance of the AB code is compared with group-based codes. The block error rate (bler), which corresponds to errors in decoding the 3×3 transmitted matrices, is demonstrated as the error event of interest. The number of receive antennas is 1. Note that the AB code has a fast decoding method while the decoding of the group-based codes and the non-group code needs exhaustive search.

In Fig. 1, we compare the AB codes at rates 2.9, 3.39, 3.53 with the group-based code $G_{171,64}$ at rate 3. We can see that at the bler of 10^{-3} , the AB code is 1dB better than the group-based code with rate 0.1 lower and is about 2dB worse than the group-based code with rates 0.39, 0.53 higher.

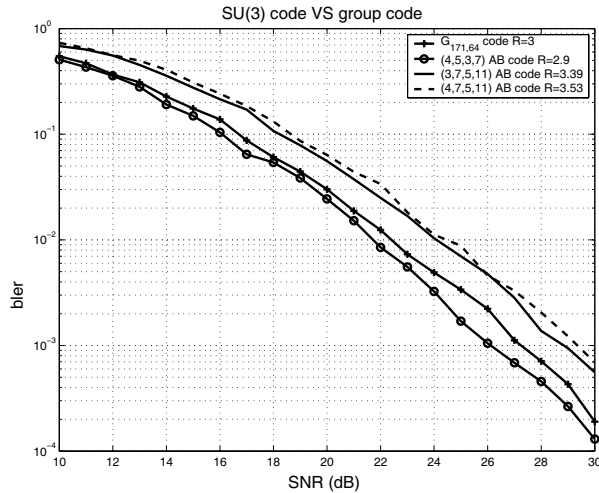


Fig. 1. Comparison of the rate 2.9, (4, 5, 3, 7) AB code, rate 3.53, (4, 7, 5, 11) AB code with $\theta = 2\pi(-\frac{p}{P} + \frac{q}{Q})$, $\xi = 2\pi(-\frac{r}{R} - \frac{s}{S})$, rate 3.39, (3, 7, 5, 11) AB code with $\theta = 2\pi(\frac{p}{P} - \frac{q}{Q})$, $\xi = 2\pi(-\frac{r}{R} - \frac{s}{S})$ with the rate 3, $G_{171,64}$ code.

In Fig. 2, we compare the AB codes at rates 3.98, 4.55 with the group-based $G_{1365,16}$ code at rate 4 and the carefully designed non-group code at rate 4. The plot shows that the AB code at rate 3.98 has a much better performance than the group-based code and is as good as the non-group code. And also, the rate 4.55 AB code still performances slightly better than the group-based code even with rate 0.55 higher.

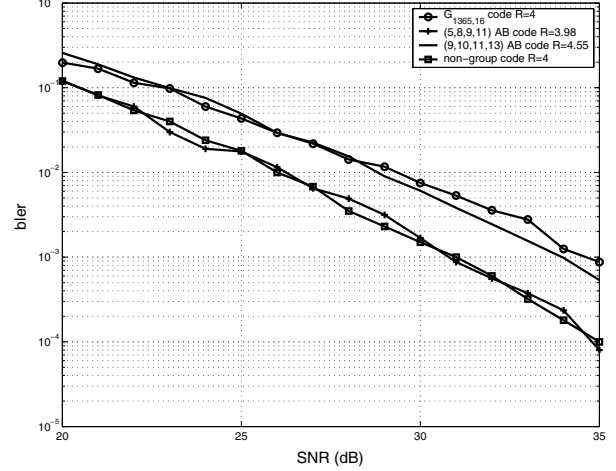


Fig. 2. Comparison of the rate 3.98, (5, 8, 9, 11) AB code, the rate 4.55, (9, 10, 11, 13) AB code with $\theta = 2\pi(-\frac{p}{P} + \frac{q}{Q})$, $\xi = 2\pi(-\frac{r}{R} - \frac{s}{S})$, with the rate 4 $G_{1365,16}$ code and non-group code.

6. REFERENCES

- [1] A. Shokrollahi, B. Hassibi, B. Hochwald, and W. Sweldens, "Representation theory for high-rate multiple-antenna code design," *IEEE Trans. Info. Theory*, vol. 47, no. 6, pp. 2335–2367, Sep. 2001.
- [2] B. Hochwald and W. Sweldens, "Differential unitary space time modulation," *IEEE Trans. Comm.*, vol. 48, pp. 2041–2052, Dec. 2000.
- [3] B. Hassibi and M. Khorrami, "Fully-diverse multi-antenna constellations and fixed-point-free Lie groups," *submitted to IEEE Trans. Info. Theory*, 2000.
- [4] Y. Jing and B. Hassibi, "Design of fully-diverse multiple-antenna codes based on $sp(2)$," *Submitted to IEEE Trans. Info. Theory*, 2003.
- [5] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE J. Sel. Area Comm.*, pp. 1451–1458, Oct. 1998.
- [6] B.M. Hochwald and S. ten Brink, "Achieving near-capacity on a multiple-antenna channel," *IEEE Trans. on Communications*, vol. 51, pp. 389–399, Mar. 2003.
- [7] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. Info. Theory*, vol. 45, pp. 139–157, 1999.
- [8] B. M. Hochwald and T. L. Marzetta, "Unitary space-time modulation for multiple-antenna communication in Rayleigh flat-fading," *IEEE Trans. Info. Theory*, vol. 46, pp. 543–564, Mar. 2000.
- [9] B. Hassibi and B. Hochwald, "Cayley differential unitary space-time codes," *IEEE Trans. Info. Theory*, pp. 1458–1503, 2002.
- [10] Barry Simon, *Representations of finite and compact Lie Groups*, Providence, R.I. : American Mathematical Society, 1994.